

Göteborg ITP 94-8

July 1994

hep-th/9408087

General branching functions of affine Lie algebras.

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Abstract

Explicit expressions are presented for general branching functions for cosets of affine Lie algebras \hat{g} with respect to subalgebras \hat{g}' for the cases where the corresponding finite dimensional algebras g and g' are such that g is simple and g' is either simple or sums of $u(1)$ terms. A special case of the latter yields the string functions. Our derivation is purely algebraical and has its origin in the results on the BRST cohomology presented by us earlier. We will here give an independent and simple proof of the validity of our results. The method presented here generalizes in a straightforward way to more complicated g and g' such as e g sums of simple and $u(1)$ terms.

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Since the discovery of Kac-Moody algebras [1],[2] and the centrally extended affine ones, known as affine Lie algebras (of which a first example was given in [3]), affine Lie algebras have played an increasingly important rôle in physics. In the framework of conformal field theories, affine Lie algebras appear in the study of string theory as well as critical phenomena in solid state theory. In particular, the so-called coset construction is crucial for the description of known conformal field theories using affine Lie algebras. Examples of this construction first appeared in [3] although a general form of the stress-energy tensor was first given by [4]. It is in this connection important to find the partition function describing the different cosets. In string theory they are nothing but the zero-point one-loop amplitudes. A slightly more general problem is to find the complete *branching functions* of the different cosets, since from the knowledge of the latter the partition functions are easily derived.

The purpose of this note is to present general expressions of the branching functions of \hat{g} with respect to a subalgebra \hat{g}' in the cases when the finite dimensional algebras g and g' are such that g is simple and g'

is either simple or a sum of $u(1)$ terms. In the latter case we can specialize to the Cartan subalgebra of g . Then the branching functions are essentially the *string functions*. The technique which we will use here was first given in [5] and the present note is a rather straightforward application of this. However, we feel that the importance of the branching and string functions motivates a derivation of the explicit form for these. We will here also present an alternative and very direct proof of the correctness of the general expression. The proof in [5] is based on the computation of the cohomology of the BRST operator and, although correct, it is not very transparent for the present application.

The explicit expressions for the branching and string functions presented here are restricted solely to the cases of *integrable representations*. The basic tool which we use apart from the construction in [5] is the Weyl-Kac formula [6] of the character of an integrable representation of an affine Lie algebra. Expressions of branching and string functions in special cases have appeared previously in the literature in particular in [7] and [8] (most of the results are reviewed in [9]). The general form of the string function for the case of $\hat{su}(2)_k$ was first derived in [7]. In ref.[10] expressions for the string functions of simply laced algebras as well as branching functions for several coset constructions was presented. The basic assumption in this work is the existence of a free field resolution of the irreducible affine module. Such

a resolution has only been proven to exist for the case of $\hat{su}(2)$ [11]. The expressions derived here for the string functions will confirm the results given in [10]. In ref. [12] a general formula for branching functions in terms of string functions was derived³. Using the explicit formulas for the branching as well as the string functions given here, one may verify these relations. We will, however, use our methods to give a simple derivation of the branching functions directly in terms of string functions. These expressions do not directly coincide with ref.[12], but we expect that some straightforward algebraic rearrangements will make them equal. One may consider more general coset constructions than the ones discussed here. The cases where instead of a simple subalgebra g' one has several simple terms $g'^{(1)}, g'^{(2)}, \dots$, yield branching functions which are easily derived using the methods presented here. This is true also in the case where one or more of the simple terms are replaced with $u(1)$ terms. By taking sums of several simple algebras $g^{(i)}$, one may have many different cases. In [13] a number of such cases will be presented: $\hat{g} = \hat{g}_{k_1} \oplus \hat{g}_{k_2}$ and $\hat{g}' = \hat{g}_{k_1+k_2}$, \hat{g} and \hat{g}' with $\text{rank } g = \text{rank } g'$, $\hat{g} = (\oplus_{a=1}^n \hat{g}_{k_a})$ and $\hat{g}' = \hat{g}_{(\sum_{i=1}^n k_i)}$, $\hat{g} = \hat{g}_{k_1} \oplus \hat{g}_{k_2}'$ and $\hat{g}' = \hat{g}_{k_1+k_2}''$, where $g'' \subseteq g$.

Let g be a simple finite dimensional Lie algebra and g' be a subalgebra of g . Their ranks are r and r' . We denote by \hat{g} and \hat{g}' the corresponding affine Lie algebras. The levels k and k' of \hat{g} and \hat{g}' are taken to be non-negative integers ($k = k'$ for g' simple and g simply laced). The set of roots of the finite algebra g are $\alpha \in \Delta_g$ and $\alpha' \in \Delta_{g'}$ for g' . The corresponding affine roots are $\hat{\alpha} \in \Delta_{\hat{g}}$ etc. The highest root of g is denoted by ψ_g and its length is taken to be one. The restrictions to positive roots are denoted Δ_g^+ and $\Delta_{g'}^+$. The number of elements in these sets are $|\Delta_g^+|, |\Delta_{g'}^+|$, respectively. We also define $\Delta_{g,g'}^+ = \{\alpha \mid \alpha \in \Delta_g^+, \alpha \notin \Delta_{g'}^+\}$ with the number of elements $|\Delta_{g,g'}^+| = |\Delta_g^+| - |\Delta_{g'}^+|$. The weight and root lattices of g and g' are denoted by $\Gamma_{w,g}, \Gamma_{r,g}, \Gamma_{w,g'}$ and $\Gamma_{r,g'}$ etc for \hat{g} and \hat{g}' . Define $\rho \in \Gamma_{r,g}$ by $\{\rho \mid \alpha_i \cdot \rho = \alpha_i^2, \text{ for simple roots } \alpha_i \in \Delta_g\}$. $\rho', \hat{\rho}$ and $\hat{\rho}'$ are the corresponding vectors for g', \hat{g} and \hat{g}' . For the finite dimensional algebras ρ is just the sum of positive roots.

Let $P_{\hat{g}}^+$ and $P_{\hat{g}'}^+$ be the set of integrable highest weight representations of \hat{g} and \hat{g}' with respective highest weights $\hat{\lambda} \in \Gamma_{w,\hat{g}}$ and $\hat{\lambda}' \in \Gamma_{w,\hat{g}'}$. Then $P_{\hat{g}}^+ = \{\hat{\lambda} \mid \hat{\alpha}_i \cdot \hat{\lambda} \geq 0 \text{ for simple roots } \hat{\alpha}_i \in \Delta_{\hat{g}}\}$, or equivalently $P_{\hat{g}}^+ = \{\lambda \mid \alpha_i \cdot \lambda \geq$

³We thank V.G. Kac for this reference.

0 for simple roots $\alpha_i \in \Delta_g$ and $\psi_g \cdot \lambda \leq k/2$ }. Let $e_{\hat{\alpha}}$, $f_{\hat{\alpha}}$ and \hat{h}_i , $\hat{\alpha} \in \Delta_{\hat{g}}^+$ and $i = 0, 1, \dots, r$, be a realization of \hat{g} according to the triangular decomposition $\hat{g} = \hat{n}_+ \oplus \hat{n}_- \oplus \hat{h}$ and $e_{\hat{\alpha}'}$, $f_{\hat{\alpha}'}$ and $\hat{h}_{i'}$, $i' = 0, 1, \dots, r'$, be a corresponding realization of \hat{g}' . \hat{h}_i , $i = 1, \dots, r$ and \hat{h}'_i , $i = 1, \dots, r'$ provide realizations of the Cartan subalgebras h and h' of g and g' . It is assumed throughout for all g' that $h' \subseteq h$. We will use the notation ν_{\parallel} for the components of ν in h' and ν_{\perp} for the remaining components. Correspondingly, we will decompose the scalar product $\nu \cdot \theta = (\nu \cdot \theta)_{\parallel} + (\nu \cdot \theta)_{\perp}$.

Let $L^g(\hat{\lambda})$ be an irreducible \hat{g} -module of highest weight $\hat{\lambda}$. We define the character of $L^g(\hat{\lambda})$ ⁴

$$\chi_{\hat{\lambda}}^g(q, \theta) = \text{Tr}_{L^g(\hat{\lambda})} \left(q^{L_0} e^{i\theta \cdot h} \right). \quad (1)$$

Here the trace is taken over all the vectors in $L^g(\hat{\lambda})$ and L_0 is the zero mode of the Virasoro-generators of \hat{g} in the Sugawara construction. In the case where $\hat{\lambda} \in P_{\hat{g}}^+$ the character is given by the Weyl-Kac formula [6],

$$\chi_{\hat{\lambda}}^g(q, \theta) = \sum_{t \in \Lambda_{r,g}^{\vee}} q^{\frac{(\lambda + \rho/2 + t)^2 - \rho^2/4}{k + c_g}} \sum_{w \in W_g} \epsilon(w) e^{i(w(\lambda + \rho/2 + t) - \rho/2) \cdot \theta} R_g^{-1}(q, \theta), \quad (2)$$

where $\Lambda_{r,g}^{\vee}$ denotes a lattice which is spanned by $t = n(k + c_g)\alpha/\alpha^2$, where $\alpha \in \Delta_g^+$, $n \in \mathbb{Z}$, and c_g is the second Casimir of the adjoint representation of g (c'_g will denote the value for g'). w are elements of the Weyl group W_g and

$$R_g(q, \theta) = \prod_{n=1}^{\infty} (1 - q^n)^r \prod_{\alpha \in \Delta_g^+} (1 - q^n e^{i\alpha \cdot \theta}) (1 - q^{n-1} e^{-i\alpha \cdot \theta}). \quad (3)$$

Define now the *coset module* $L^{g,g'}(\hat{\lambda}, \hat{\lambda}')$ by

$$L^{g,g'}(\hat{\lambda}, \hat{\lambda}') = \{v \in L^g(\hat{\lambda}) \mid e_{\hat{\alpha}'}(v) = 0 \text{ and } \hat{h}'_i(v) = \hat{\lambda}'_i v \text{ for } \hat{\alpha}' \in \Delta_{\hat{g}'}^+ \text{ and } \hat{h}'_i \in \hat{h}'\}. \quad (4)$$

The branching function of this coset module is then defined as:

$$b_{\hat{\lambda}, \hat{\lambda}'}^{g,g'}(q, \theta_{\perp}) = \text{Tr}_{L^{g,g'}(\hat{\lambda}, \hat{\lambda}')} \left(q^{L_0^{g,g'}} e^{i(\theta \cdot h)_{\perp}} \right). \quad (5)$$

We have here introduced the notation $L_0^{g,g'} = L_0^g - L_0^{g'}$. The definition of the branching function implies, in the case of integrable representations, that the character of

⁴We are omitting the conventional factor $q^{-c/24}$ throughout this work.

$L^g(\hat{\lambda})$ decomposes as

$$\chi_{\lambda}^g(q, \theta) = \sum_{\lambda' \in P_{\hat{g}'}^+} b_{\lambda, \lambda'}^{g, g'}(q, \theta_{\perp}) \chi_{\lambda'}^{g'}(q, \theta_{\parallel}). \quad (6)$$

The main fact needed to show that eq.(5) implies eq. (6) is that $L^g(\hat{\lambda})$ is isomorphic as a \hat{g}' -module to a direct sum of modules $L^{g'}(\hat{\lambda}')$ with $\hat{\lambda}' \in P_{\hat{g}'}^+$ (see [7], §4.9).

In [5] it was proven that the coset module $L^{g, g'}$ is isomorphic to the cohomology group $\ker \hat{Q} / \text{Im} \hat{Q}$ of the so-called relative BRST operator \hat{Q} acting on the module $M^g(\hat{\lambda}) \otimes M^{\tilde{g}}(\hat{\lambda}') \otimes M^{gh}$. Here $M^g(\hat{\lambda})$ is the \hat{g} highest weight Verma module and $M^{\tilde{g}}(\hat{\lambda}')$ is a $\hat{\tilde{g}}$ Verma module, where $\tilde{g} = g'$ and $\tilde{k}' = -k' - 2c'_g$ and finally M^{gh} is a module originating from the Faddeev-Popov ghosts, which have been introduced in the BRST approach. The isomorphism holds under the assumption that $\hat{\lambda} \in P_{\hat{g}}^+$ and $-\hat{\lambda}' - \rho' \in P_{\hat{g}'}^+$. We will not explain the details of the construction and proof of the isomorphism here, but refer to [5]. One should note, however, that the restriction on the highest weights $\hat{\lambda}'$ implies that $M^{\tilde{g}}(\hat{\lambda}')$ is irreducible, as can be seen from the Kac-Kazhdan determinant [14]. Instead we proceed more directly by giving the implications of the isomorphism for the branching functions. First we define the following characters

$$\begin{aligned} \tilde{\chi}_{\lambda'}^{g'}(q, \theta_{\parallel}) &\equiv e^{i\tilde{\lambda}' \cdot \theta_{\parallel}} q^{-\frac{\tilde{\lambda}' \cdot (\tilde{\lambda}' + \rho')}{k' + c'_g}} R_{g'}^{-1}(q, \theta_{\parallel}), \quad -\tilde{\lambda}' - \rho' \in P_{\hat{g}'}^+ \\ \chi^{gh}(q, \theta_{\parallel}) &\equiv e^{i\rho' \cdot \theta_{\parallel}} R_{g'}^2(q, \theta_{\parallel}) \end{aligned} \quad (7)$$

for g' simple and

$$\begin{aligned} \tilde{\chi}_{\lambda'}^{g'}(q, \theta_{\parallel}) &\equiv e^{i\tilde{\lambda}' \cdot \theta_{\parallel}} q^{-\sum_{i=1}^{r'} \frac{\tilde{\lambda}'_i^2}{k'_i}} \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{r'}} \\ \chi^{gh}(q, \theta_{\parallel}) &\equiv \prod_{n=1}^{\infty} (1 - q^n)^{2r'} \end{aligned} \quad (8)$$

for $g' = u(1) \oplus u(1) \oplus \dots$ (r' terms). Here k'_i are the levels of $\hat{su}(2)$ such that the $u(1)$ terms are embedded as Cartan elements in the corresponding finite dimensional algebra ($k'_i = k$ for g simply laced). The characters are nothing but the characters of $M^{g'}(\tilde{\lambda}')$ and M^{gh} as discussed in [5]. We introduce also a formal integration of exponents defined by⁵ $\int d^{r'} \theta_{\parallel} e^{i\theta_{\parallel} \cdot \nu} \equiv \prod_{i=1}^{r'} \delta_{\nu_i, 0} \equiv \delta_{\nu, 0}$.

⁵One may explicitly represent this formal integration as the limit $L \rightarrow \infty$ of $\frac{1}{L^{r'}} \int_{-L/2}^{L/2} d^{r'} \theta_{\parallel}$.

PROPOSITION 1. *With the notation above and g' simple or $g' = u(1) \oplus u(1) \oplus u(1) \dots$ (r' terms) we have*

$$b_{\lambda, \lambda'}^{g, g'}(q, \theta_{\perp}) = \int d^{r'} \theta_{\parallel} \chi_{\lambda}^g(q, \theta) \tilde{\chi}_{\lambda'}^{g'}(q, \theta_{\parallel}) \chi^{gh}(q, \theta_{\parallel}) \quad (9)$$

where $\hat{\lambda}' \equiv -\hat{\lambda}' - \hat{\rho}' \in P_{\hat{g}'}^+$, $\hat{\lambda} \in P_{\hat{g}}^+$.

Proof. Using eq.(6) the right-hand side of eq. (9) reads

$$\sum_{\lambda' \in P_{\hat{g}'}^+} b_{\lambda, \lambda'}^{g, g'}(q, \theta_{\perp}) \int d^{r'} \theta_{\parallel} \chi_{\lambda'}^{g'}(q, \theta_{\parallel}) \tilde{\chi}_{\lambda'}^{g'}(q, \theta_{\parallel}) \chi^{gh}(q, \theta_{\parallel}). \quad (10)$$

since the branching function is independent of the variables integrated over. For g' simple we use the Weyl-Kac formula eq.(2) for $\chi_{\lambda'}^{g'}$ and eq.(7) to perform the integration. The right-hand side of eq.(9) is then

$$\sum_{\lambda' \in P_{\hat{g}'}^+} b_{\lambda, \lambda'}^{g, g'}(q, \theta_{\perp}) \sum_{t' \in \Lambda_{r, g'}^{\vee}} \sum_{w \in W_{g'}} \epsilon(w) q^{\frac{(\lambda' + \rho'/2 + t')^2 - \rho'^2/4}{k' + c_g'}} q^{-\frac{\bar{\lambda}' \cdot (\bar{\lambda}' + \rho')}{k' + c_g'}} \delta_{[w(\lambda' + \rho'/2 + t') - \rho'/2 + \bar{\lambda}' + \rho'], 0}.$$

For $\hat{\mu}', \hat{\nu}' \in P_{\hat{g}'}^+$ and $w \in W_{\hat{g}'}$, the affine Weyl group of g' , we have that the equation $w(\hat{\mu}' + \hat{\rho}'/2) - \hat{\rho}'/2 = \hat{\nu}'$ implies that $\hat{\mu}' = \hat{\nu}'$ and $w = Id$. (for a proof in the finite dimensional case see *e.g.* [15], lemma A in section 13.2 and for the affine case [9], lemma 10.3 together with prop. 3.12b). Then

$$\sum_{t' \in \Lambda_{r, g'}^{\vee}} \sum_{w \in W_{g'}} \epsilon(w) q^{\frac{(\lambda' + \rho'/2 + t')^2 - \rho'^2/4}{k' + c_g'}} q^{-\frac{\bar{\lambda}' \cdot (\bar{\lambda}' + \rho')}{k' + c_g'}} \delta_{[w(\lambda' + \rho'/2 + t') - \rho'/2 + \bar{\lambda}' + \rho'], 0} = \delta_{\lambda' + \bar{\lambda}' + \rho', 0}$$

which proves the proposition for the case of g' being simple. The case $g' = u(1) \oplus u(1) \oplus \dots \oplus u(1)$ is completely analogous. \square

THEOREM. *Let g be a simple finite dimensional algebra, \hat{g} its affine extension, $\hat{\lambda} \in P_{\hat{g}}^+$ the highest weight of a representation of \hat{g} , \hat{g}' a subalgebra of \hat{g} with $h' \subseteq h$ and $\hat{\lambda}' \in P_{\hat{g}'}^+$. Then the affine branching function $b_{\lambda, \lambda'}^{g, g'}(q, \theta_{\perp})$ is*

(i) g' simple:

$$b_{\lambda, \lambda'}^{g, g'}(q, \theta_{\perp}) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{r - r' + 2|\Delta_{g, g'}^+|}} \sum_{t \in \Lambda_{r, g}^{\vee}} \sum_{w \in W_g} \epsilon(w) q^{\frac{(\lambda + \rho/2 + t)^2 - \rho^2/4}{k + c_g}} q^{-\frac{(\lambda' + \rho'/2)^2 - \rho'^2/4}{k' + c_g'}}$$

$$e^{i((w(\lambda+\rho/2+t)-\rho/2)\cdot\theta)_\perp} \prod_{\alpha \in \Delta_{g,g'}^+} \sum_{p_\alpha} \sum_{s_\alpha=0}^{\infty} \left((-1)^{s_\alpha} q^{\frac{1}{2}[(s_\alpha-p_\alpha+1/2)^2-(p_\alpha-1/2)^2]} e^{ip_\alpha(\alpha\cdot\theta)_\perp} \right) \quad (11)$$

where $p_\alpha \in \mathcal{Z}$ is restricted by

$$\sum_{\alpha \in \Delta_{g,g'}^+} p_\alpha \alpha_\parallel + (w(\lambda + \rho/2 + t) - \rho/2)_\parallel - \lambda' = 0. \quad (12)$$

(ii) $g' = u(1) \oplus u(1) \oplus \dots$ (r' terms) :

$$b_{\lambda,\lambda'}^{g,g'}(q, \theta_\perp) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{r-r'+2|\Delta_g^+|}} \sum_{t \in \Lambda_{r,g}^\vee} \sum_{w \in W_g} \epsilon(w) q^{\frac{(\lambda+\rho/2+t)^2 - \rho^2/4}{k+cg} - \sum_{i=1}^{r'} \frac{\lambda_i'^2}{k_i'}} e^{i((w(\lambda+\rho/2+t)-\rho/2)\cdot\theta)_\perp} \prod_{\alpha \in \Delta_g^+} \sum_{p_\alpha} \sum_{s_\alpha=0}^{\infty} \left((-1)^{s_\alpha} q^{\frac{1}{2}[(s_\alpha-p_\alpha+1/2)^2-(p_\alpha-1/2)^2]} e^{ip_\alpha(\alpha\cdot\theta)_\perp} \right) \quad (13)$$

where $p_\alpha \in \mathcal{Z}$ is restricted by

$$\sum_{\alpha \in \Delta_g^+} p_\alpha \alpha_\parallel + (w(\lambda + \rho/2 + t) - \rho/2)_\parallel - \lambda' = 0 \quad (14)$$

Proof: We use eq.(9). In order to perform the integrations we first write

$$R_g(q, \theta) = R_{g'}(q, \theta_\parallel) \prod_{n=1}^{\infty} (1 - q^n)^{r-r'} \prod_{\alpha \in \Delta_{g,g'}^+} (1 - q^n e^{i\alpha\cdot\theta}) (1 - q^{n-1} e^{-i\alpha\cdot\theta}) \quad (15)$$

and use the identity [16]-[18]

$$\prod_{\alpha \in \Delta_{g,g'}^+} (1 - q^n e^{i\alpha\cdot\theta})^{-1} (1 - q^{n-1} e^{-i\alpha\cdot\theta})^{-1} = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{2|\Delta_{g,g'}^+|}} \prod_{\alpha \in \Delta_{g,g'}^+} \sum_{p_\alpha} \sum_{s_\alpha=0}^{\infty} (-1)^{s_\alpha} q^{\frac{1}{2}[(s_\alpha-p_\alpha+1/2)^2-(p_\alpha-1/2)^2]} e^{ip_\alpha \alpha \cdot \theta}, \quad (16)$$

where $p_\alpha \in \mathcal{Z}$ for every $\alpha \in \Delta_{g,g'}^+$, to rewrite the expression for the character of $L^g(\hat{\lambda})$ eq.(2). Then this together with eq.(7) inserted into eq.(9) yields for the case of g' simple

$$b_{\lambda,\lambda'}^{g,g'}(q, \theta_\perp) = \frac{1}{\prod_{n=1}^{\infty} (1 - q^n)^{r-r'+2|\Delta_{g,g'}^+|}} \sum_{t \in \Lambda_{r,g}^\vee} q^{\frac{(\lambda+\rho/2+t)^2 - \rho^2/4}{k+cg} - \frac{(\tilde{\lambda}'+\rho'/2)^2 - \rho'^2/4}{k'+c'_g}}$$

$$\begin{aligned}
& \sum_{w \in W_g} \epsilon(w) \int d^{r'} \theta_{\parallel} e^{i(w(\lambda + \rho/2 + t) - \rho/2) \cdot \theta} e^{i(\tilde{\lambda}' + \rho') \cdot \theta_{\parallel}} \\
& \prod_{\alpha \in \Delta_{g,g'}^+} \sum_{p_{\alpha}} \sum_{s_{\alpha}=0}^{\infty} (-1)^{s_{\alpha}} q^{\frac{1}{2}[(s_{\alpha} - p_{\alpha} + 1/2)^2 - (p_{\alpha} - 1/2)^2]} e^{ip_{\alpha} \alpha \cdot \theta}. \quad (17)
\end{aligned}$$

Upon integrating over θ_{\parallel} we will get eq.(11) and deltafunctions that impose the condition eq.(12), where $\lambda' = -\tilde{\lambda}' - \rho'$. The proof of (ii) is completely analagous. \square

The important special case of (ii) above is when the branching functions are the string functions (up to a factor η^{-r} , where η is the Dedekind function). The latter are then given by eq.(13) with $r' = r$ and read explicetly

$$\begin{aligned}
c_{\lambda, \lambda'}^g(q) &= \frac{\eta^{-r}(q)}{\prod_{n=1}^{\infty} (1 - q^n)^{2|\Delta_g^+|}} \sum_{t \in \Lambda_{r,g}^{\vee}} \sum_{w \in W_g} \epsilon(w) q^{\frac{(\lambda + \rho/2 + t)^2 - \rho^2/4}{k + c_g} - \sum_{i=1}^r \frac{\lambda_i'^2}{k_i'}} \\
& \prod_{\alpha \in \Delta_g^+} \sum_{p_{\alpha}} \sum_{s_{\alpha}=0}^{\infty} \left((-1)^{s_{\alpha}} q^{\frac{1}{2}[(s_{\alpha} - p_{\alpha} + 1/2)^2 - (p_{\alpha} - 1/2)^2]} \right), \quad (18)
\end{aligned}$$

where $p_{\alpha} \in \mathcal{Z}$ is restricted by

$$\sum_{\alpha \in \Delta_g^+} p_{\alpha} \alpha + w(\lambda + \rho/2 + t) - \rho/2 - \lambda' = 0. \quad (19)$$

Let us comment that there are completely analagous expressions for the more general case of g' consisting of a mixture of several simple or $u(1)$ terms, but we have chosen not give the expressions explicetly. They may be derived in the same straightforward manner as above.

We finally derive an expression for the branching functions in terms of the string functions.

PROPOSITION 2. *With the same notation as in the theorem, we have the branching function for the case of g' simple*

$$\begin{aligned}
b_{\lambda, \lambda'}^{g,g'}(q, \theta_{\perp}) &= \frac{q^{\frac{r}{24}}}{\prod_{n=1}^{\infty} (1 - q^n)^{|\Delta_{g'}^+| - 2r - r'}} q^{-\frac{\lambda'(\lambda' + \rho')}{k' + c'_{g'}}} \\
& \sum_{\mu \in \Gamma_{w,g}} c_{\lambda, \mu}^g(q) q^{\sum_{i=1}^r \frac{\mu_i^2}{k_i}} e^{i(\mu \cdot \theta)_{\perp}} \prod_{\alpha' \in \Delta_{g'}^+} \sum_{p_{\alpha'}} (-1)^{p_{\alpha'}} q^{\frac{1}{2}[(p_{\alpha'} + 1/2)^2 - 1/4]} \quad (20)
\end{aligned}$$

where $p_{\alpha'} \in \mathcal{Z}$ is restricted by

$$\sum_{\alpha' \in \Delta_{g'}^+} p_{\alpha'} \alpha' + \mu_{\parallel} - \lambda' = 0. \quad (21)$$

Proof: The proof of the proposition is completely analagous to that of the theorem.

However, instead of the Weyl-Kac formula for the character of \hat{g} we take eq.(6) with $g' = h$

$$\chi_{\lambda}^g(q, \theta) = \sum_{\mu \in \Gamma_{w,g}} b_{\lambda, \mu}^{g,h}(q) e^{i\mu \cdot \theta} q^{\sum_{i=1}^r \frac{\mu_i^2}{k_i}} \prod_{n=1}^{\infty} (1 - q^n)^{-r}. \quad (22)$$

Furthermore, in order to perform the integration we use the following identity in place of eq.(16)

$$R_{g'}(q, \theta_{\parallel}) = \prod_{n=1}^{\infty} (1 - q^n)^{r' - |\Delta_{g'}^+|} \prod_{\alpha' \in \Delta_{g'}} \sum_{p_{\alpha'} \in \mathcal{Z}} (-1)^{p_{\alpha'}} q^{\frac{1}{2}[(p_{\alpha'} + 1/2)^2 - 1/4]} e^{ip_{\alpha'} \alpha' \cdot \theta_{\parallel}}, \quad (23)$$

which is easily derived using the Jacobi triple product identity

$$\sum_{p=-\infty}^{\infty} q^{p^2} e^{ip\theta} = \prod_{p=1}^{\infty} (1 - q^{2p})(1 + q^{2p-1} e^{i\theta})(1 + q^{2p-1} e^{-i\theta}).$$

□

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